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ORIGINAL PAPER

Interference of two Bose–Einstein condensates with varying initial conditions

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Abstract In this work, we study the interference effect of two independent Bose–Einstein condensates by numerical technique. Symplectic algorithm with high order difference scheme is used to solve the one-dimensional (1D) Gross–Pitaevskii (GP) equation. It is shown that the property of the interference between two condensates resembles the double-slit interference, and the fringe is dependent on the relative phase and the inter-spacing of the initial clouds. Interference of condensate consisting of different number of atoms is also presented, and the fringe is shown to be unsymmetrical. The periodic evolution of the two condensates confined in a harmonic trapping potential is studied and a much stronger interference pattern is shown.

Keywords Gross–Pitaevskii equation · Bose–Einstein condensation · Interference · Symplectic method

1 Introduction

The interference of Bose–Einstein condensates (BEC) has been much studied both experimentally and theoretically for determining the coherence properties of the condensate [1–3]. It is a common procedure to produce one dimensional (1D) Bose–Einstein condensate nowadays [4,5], and it is convenient to study the interference effect in 1D BEC with a simple model wavefunction consisting of a linear combina-

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tion of two Gaussians. This wavefunction can describe many of the observed features seen in the interaction of two clouds after they released. The interference of two condensates was studied to probe the condensate as coherence state and its relative global phase [6]. Interference effect of three condensates was reported and was compared with the two condensates case, and the existence of the global phase was further demonstrated [7,8]. The dynamics of condensate in a magnetic trap and an optical lattice was studied, and the evolution of interference for an array of condensates was presented after the lattices were switched off [9]. The interference can happen in a single condensate and which was discussed explicitly [10]. Experimentaly, the Bose–Einstein condensate has recently been observed in a quasi-uniform potential and the coherence nature of the condensate has been confirmed by a matter-wave interference experiment [11].

In this work, we numerically study the interference effect between two condensates by symplectic method. The wavefunction is governed by GP equation, a self-consistent mean field equation which incorporates the interaction of atoms and the trapping potential [12]. It has a form of nonlinear Schrödinger equation and can be transformed into Hamiltonian system which has symplectic structure. Symplectic method can preserve the symplecic structure of the nonlinear Hamiltonian system and the normalization of the wavefuction is naturally preserved in computation [13–16]. It is superior to some other method in long-time and many-step computation.

In Sect. 2, we study the time-dependent 1D GP equation, transform it into dimensionless form and discrete it symplectically. The ground state wavefunction is obtained by symplectic method and the normalization of the wavefunction is tested. In Sect. 3, we discuss the interference pattern of two condensates of different relative phase and of different initial inter-spacing. The interference of two condensates with different atom numbers is also discussed in this section. In Sect. 4 we present the interference of two condensates confined in a harmonic trapping potential, periodic nature of the condensates evolution is shown and the interaction of the condensates is discussed. Finally conclusion is given in Sect. 5.

2 The time-dependent 1D GP equation and symplectic method

The 1D time-dependent GP equation with harmonic trapping potential can be written as,

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega^2x^2+\lambda_{1D}\left|\psi(x,t)\right|^2\right]\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}, \tag{1}$$

with $\lambda_{1D}=\frac{4\pi\hbar^2 a_s N}{ma^2}$ represents the interaction between atoms. Here m is the mass of a single atom, ω is the frequency of the harmonic trapping potential, N is the total number of atoms in the condensate, and a_s is the scattering length. $a=\sqrt{\hbar/2m\omega}$ is the harmonic oscillator length. Equation (1) can be rescaled into dimensionless form

$$\left[-\frac{\partial^2}{\partial\xi^2}+\frac{\xi^2}{4}+\alpha\left|\Phi(\xi,\tau)\right|^2\right]\Phi(\xi,\tau)=i\frac{\partial\Phi(\xi,\tau)}{\partial\tau} \tag{2}$$

with $\xi = x/a$, $\tau = \omega t$, then $\Phi(\xi) = \psi(x)\sqrt{a}$, and the nonlinear coefficient becomes $\alpha = 8\pi N a_s/a$. The normalization condition is

$$I = \int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx = \int_{-\infty}^{\infty} |\Phi(\xi,\tau)|^2 \, d\xi = 1 \tag{3}$$

If we write the wavefunction in terms of its real and imagine parts separately as $\Phi(\xi,\tau) = u(\xi,\tau) + iv(\xi,\tau)$, then Eq. (2) becomes

$$\begin{aligned} \dot{u} &= -v_{\xi\xi} + \frac{\xi^2}{4} v + \alpha(u^2+v^2)v \\ \dot{v} &= -\left[-u_{\xi\xi} + \frac{\xi^2}{4} u + \alpha(u^2+v^2)u\right] \end{aligned}. \tag{4}$$

In order to get high accuracy we adopt 4-order central space difference [17,18]

$$\begin{aligned} \frac{\partial^2 u}{\partial \xi^2} &\approx \frac{1}{12h^2}[-u_{j-2} + 16u_{j-1} - 30u_j + 16u_{j+1} - u_{j+2}] \\ \frac{\partial^2 v}{\partial \xi^2} &\approx \frac{1}{12h^2}[-v_{j-2} + 16v_{j-1} - 30v_j + 16v_{j+1} - v_{j+2}] \end{aligned} \tag{5}$$

to discrete equation (4) in space, $j = 0, 1, 2, \ldots N$, and h is the space step size. Zero boundary condition is considered. It can be verified that Eq. (4) can be written into Hamiltonian canonical equation

$$\dot{z} = J^{-1}\frac{\partial H}{\partial z}, \quad z = (u_0 \cdots u_N \; v_0 \cdots v_N)^T \tag{6}$$

with the hamiltonian

$$\begin{aligned} H = \frac{1}{24h^2} \sum_{j=1}^{N-1} &\big[[-u_{j-2} + 16u_{j-1} - 30u_j + 16u_{j+1} - u_{j+2}] \\ &+ [-v_{j-2} + 16v_{j-1} - 30v_j + 16v_{j+1} - v_{j+2}]\big] \\ &+ \sum_{j=1}^{N-1} \left[\frac{\xi_j^2}{8}(u_j^2 + v_j^2) + \frac{\alpha}{4}(u_j^4 + v_j^4) + \frac{\alpha}{2} u_j^2 v_j^2\right] \end{aligned} \tag{7}$$

where $J = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}$ is the standard symplectic matrix. Therefore, we can apply symplecic algorithm in solving Eq. (6), such as the symplectic Euler-center scheme

$$\frac{z^{n+1} - z^n}{\Delta\tau} = J\left(\frac{\partial H}{\partial z}\right)_{(z^{n+1}+z^n)/2}, \tag{8}$$

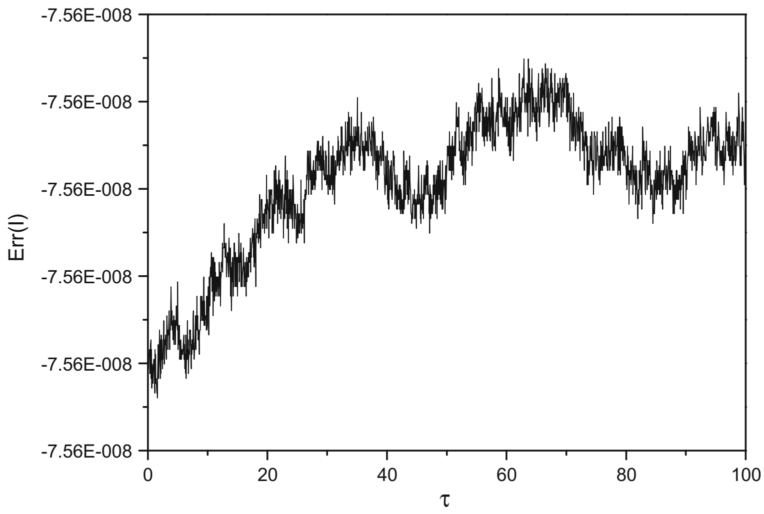

Fig. 1 Evolution of the error of normalization of the wavefunction, $\alpha = 8\pi$

which has quadratic invariant, or the symplectic Runge–Kutta scheme

$$
\begin{aligned}
z^{n+1} &= z^n + \frac{\Delta\tau}{2}(f(Y_1) + f(Y_2)) \\
Y_1 &= z^n + \Delta\tau\left(\frac{1}{4}f(Y_1) + \left(\frac{1}{4} - \frac{1}{6}\sqrt{3}\right)f(Y_2)\right) \\
Y_2 &= z^n + \Delta\tau\left(\left(\frac{1}{4} + \frac{1}{6}\sqrt{3}\right)f(Y_1) + \frac{1}{4}f(Y_2)\right)
\end{aligned}
\tag{9}
$$

which is a fourth-order symplectic scheme, here $\Delta\tau$ is the time step size.

In this paper, we adopt the Runge–Kutta scheme (9) in computation. Follow the procedure proposed in Ref. [19], we obtain the ground state eigenvalue and the corresponding wavefunction of the condensate. In our computation, space step $h = 0.1$, and time step $\Delta\tau = 0.001$. When $\alpha = 8\pi$, the eigenvalue is 4.51, and the wavefunction is tested to be stable. The conservation of the normalization of the wavefunction is tested. In Fig. 1, we plot the error of the norm Err(I) which is preserved to be 10^{-8}, and it assures the effective of the numerical scheme we used.

3 Interference between two condensates

In this section we study numerically the interference effect of two condensates. We consider $^{133}C_s$, $m = 2.2 \times 10^{-25}$ kg, the scattering length $a_s = 3.45$ nm, $\omega = 2\pi \times 10$ Hz [20], then the harmonic oscillator length $a = 1.95$ µm. The condensates are separated by a distance d and initially centered at $\xi_0 = \pm d/2$. With the wavefunction of $\alpha = 8\pi$, we translate it to present the two condensates $\Phi_L(\xi, \tau)$ and $\Phi_R(\xi, \tau)e^{i\theta}$, where θ is the relative phase between the condensates. The interaction of the two condensates

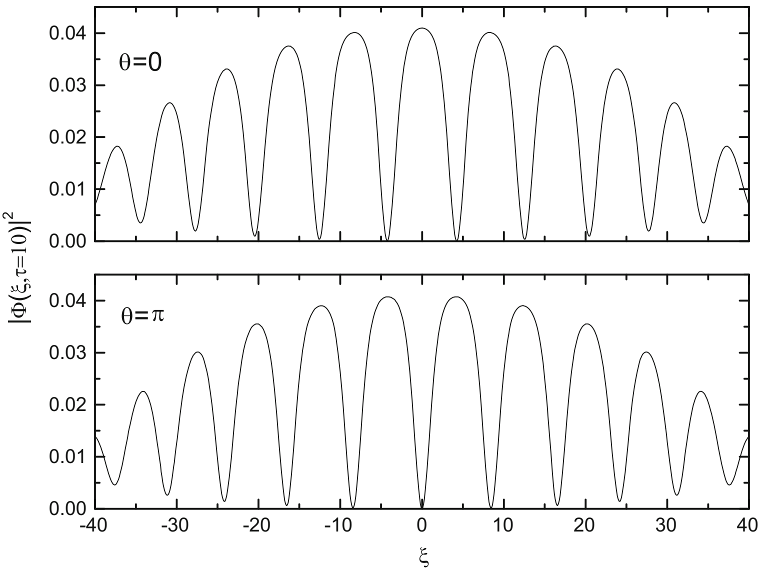

Fig. 2 The interference patterns of two condensates at $\tau = 10$ with different relative phase $\theta = 0$ and $\theta = \pi$. $d = 20, \alpha = 8\pi$

can be described by the linear combination $\Phi(\xi, \tau) = \Phi_L(\xi, \tau) + \Phi_R(\xi, \tau)e^{i\theta}$ which satisfy

$$\left[-\frac{\partial^2}{\partial \xi^2} + \alpha \left|\Phi(\xi, \tau)\right|^2\right] \Phi(\xi, \tau) = i\frac{\partial \Phi(\xi, \tau)}{\partial \tau} \tag{10}$$

when the trap is removed. The time-dependent probability of the condensates can be described by

$$\rho(\xi, \tau, \theta) = \left|\Phi_L(\xi, \tau) + \Phi_R(\xi, \tau)e^{i\theta}\right|^2. \tag{11}$$

The four order symplectic Runge–Kutta scheme (9) is applied in solving Eq. (10).

In comparison with the interference of two condensates in Ref. [6], we mainly discuss the effect of the relative phase θ and initial spacing d of the condensates. Firstly we examine two condensates initially spaced at $\xi_0 = \pm 10, d = 20$. At $\tau = 0$, the trapping potential is removed and the condensates expand, overlap, and interfere. The corresponding interference pattern at time $\tau = 10$ is plotted in Fig. 2 for different relative phases $\theta = 0$ and $\theta = \pi$. The patterns are symmetric about $\xi = 0$ and complementary between. The visibility of the central interference fringes is the best. The fringe spacing is about 16 μ m. It shows that the position of the maximum interference fringe is different for $\theta = 0$ and $\theta = \pi$, which indicates that the interference of condensates is dependent on the relative phase between them.

Secondly we examine the influence of the initial spacing d on the interference. The patterns of two condensates with different initial spacing $d = 20$ and $d = 40$

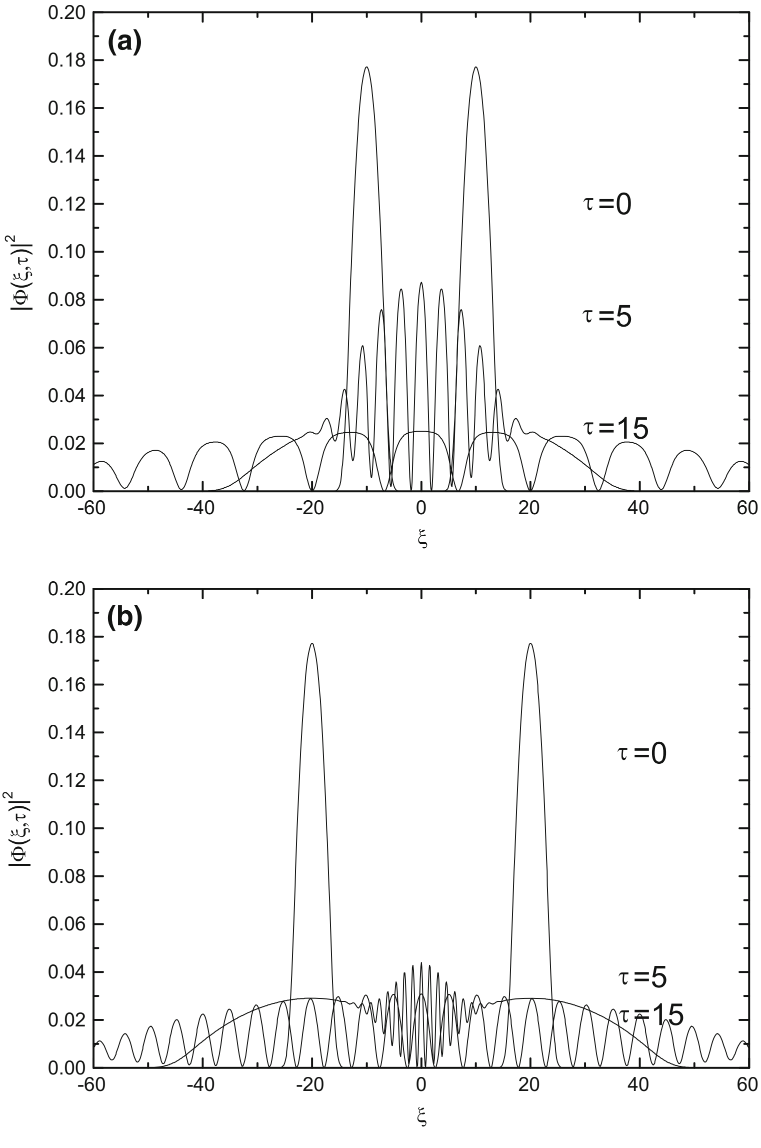

Fig. 3 The evolutions of interference patterns of two condensates. $\theta = 0$, $\alpha = 8\pi$ (**a**) $d = 20$, (**b**) $d = 40$

are compared in Fig. 3a, b, $\theta = 0$. The condensates expand, overlap, and interfere symmetrically. Through comparison we can see that at time $\tau = 15$, the fringe spacing in Fig. 3b for $d = 40$ is narrower than that of Fig. 3a for $d = 20$. That is, broader initial spacing of two condensates results in narrower interference fringe spacing and this feature resembles the law of double-slit interference of the monochromatic light.

Reference [11] realized BEC of atomic gas in quasi-uniform potential, and a interference experiment between a main condensate and a small one is done to confirm the coherence nature. Motivated by this work, we discuss the harmonic potential

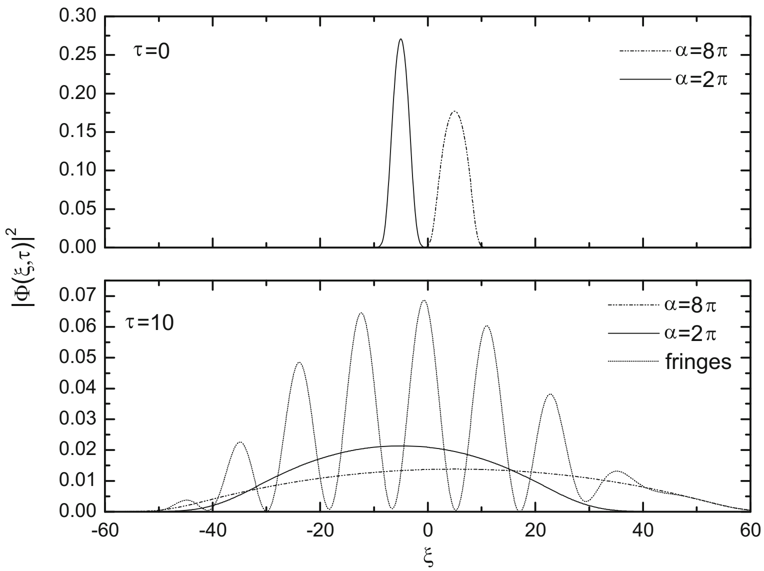

Fig. 4 Interference pattern of two condensates with different number of atoms together with independent evolution of each condensate. $d = 10$, $\theta = 0$. *Solid line* for the condensate of $\alpha = 2\pi$, *dash dot dot line* for $\alpha = 8\pi$, and *short dot line* for the interference fringes at time $\tau = 10$. The trapping potential is removed at $\tau = 0$

case which is not uniform potential, and study the interference between two BECs consisting of different number of atoms. Recall that $\alpha = 8\pi Na_s/a$, we consider a condensate of $\alpha = 8\pi$ to be the large one and another of $\alpha = 2\pi$ to be the small one, then the ratio of the number of atoms in the condensates is 4:1, the difference in atoms numbers is large. The two condensates are close to each other, they initially spaced at $\xi_0 = \pm 5, d = 10$. The trapping potential is set zero at time $\tau = 0$, and the condensates expand, overlap, and interfere. A numerical simulation is done and the result is given in Fig. 4. The interference pattern at time $\tau = 10$ is given together with the independent evolution of each condensate. It shows that the large condensate expands faster than the small one. At time $\tau = 10$, the position of the maximum central fringe is $\xi = -0.7$, and the whole interference fringes is not symmetric about it. We also find that larger distance d will result in shorter fringe spacing, and weaken the unsymmetrical property of the interference pattern at the same time. The interference pattern between two condensates consisting of different number of atoms further demonstrates the existence of the global phase.

4 Interaction of two condensates in a harmonic trapping potential

An existing of trapping potential will bring more interesting phenomenon [20]. In this section we will discuss the interaction of two condensates in a harmonic trapping potential. The two condensates are prepared and then transported symmetrically into a harmonic well, $d = 20$, and the harmonic well is not removed during the examination. The dynamic evolution of the condensates is given in Fig. 5, from which we can see

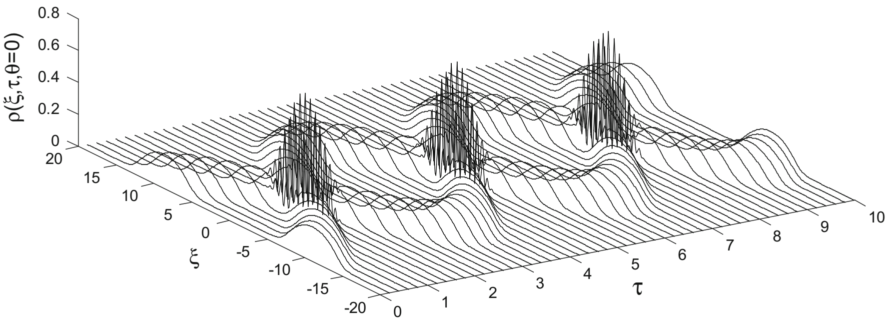

Fig. 5 The evolution of two condensates confined in a harmonic trapping potential. $d = 20$, $\theta = 0$

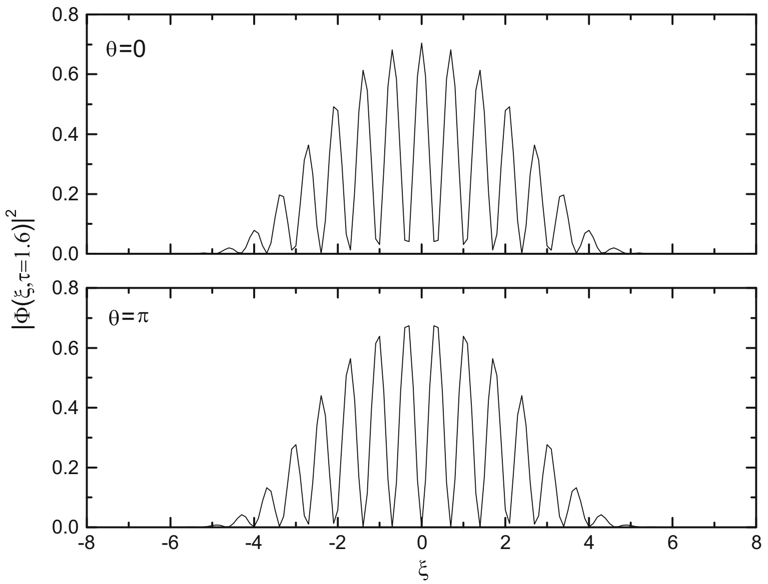

Fig. 6 The interference patterns of two condensates with $d = 20$ confined in a harmonic trapping potential at time $\tau = 1.6$ with $\theta = 0$ and $\theta = \pi$ respectively

that there is a periodic nature of the evolution. The condensates perform periodic oscillation around the minimum of the harmonic trap with a period $T = 6.288$. Since $\alpha > 0$, the interaction between the atoms is repulsive, the period is larger than 2π. The condensates evolved in the trapping potential without expansion, and interference can be seen in their overlapping region.

The interference is much stronger due to the confinement of the trapping potential as shown in Fig. 6, where we plot the interference fringes at time $\tau = 1.6$ when the two condensates fully overlap with $\theta = 0$ and $\theta = \pi$ respectively. The existence of a harmonic trapping potential greatly enhanced the visibility of the interference pattern which can be of use in experimental observation. The position of the interference fringe is complementary for the case of $\theta = 0$ and $\theta = \pi$, which means that the interference pattern is still influenced by the relative phase when the condensates are confined.

In sum, the condensates evolve periodically in the harmonic potential, interfere with each other when they overlap, and maintain their shape after they passed through each other, just like the interaction of two solitons [21].

5 Conclusion

In this work we solve the 1D time-dependent GP equation with symplectic method. Four order Runge–Kutta scheme is used accompanied by high order central space difference scheme, and the result is proved to be effective. The interference of two condensates is studied, and it shows that the interference pattern is dependent on the relative phase and the inter-spacing of the initial condensates in a manner as the law of double-slit interference. Unsymmetrical interference pattern is shown for the interference between a small condensate and a large one. We also examine the interaction of two condensates confined in a harmonic trapping potential, periodic behavior of the condensates is demonstrated and the interference in the overlapping region is much stronger due to the confinement of the external potential.

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